Universal criterion and amplitude equation for a nonequilibrium Ising-Bloch transition

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We identify a universal criterion for the onset of a nonequilibrium Ising-Bloch (NIB) transition, and describe the behavior near the bifurcation by a generic amplitude equation. We found that a NIB transition is caused by an antisymmetric eigenvector passing the translational mode of the system at a critical point. In this context we discuss Hamiltonian and dissipative systems. We report on a NIB in nonlinear optics, manifesting itself in a transition from static to moving polarization fronts.

Nonlinear systems can exhibit spatially homogeneous, periodic, or localized structures with nontrivial dynamical behavior. One of the basic issues in nonlinear physics is to correctly predict the interaction of such solutions where they coexist. For example, bistable systems can exhibit fronts connecting two stable homogeneous states. The properties and dynamics of such fronts have attracted attention across many branches of science, including chemistry, biology, fluid dynamics, and optics [1].

Generally, a front connecting two nonequivalent homogeneous states moves in such a way that the more stable state annihilates the other. Sometimes, as a consequence of a discrete symmetry, a system may possess two *equivalent* states. The front between such states is generally at rest due to symmetry. But such fronts can destabilize via a bifurcation on changing a system parameter. A prominent example among gradient systems [1] is the so-called Ising-Bloch transition [2,3], known from the physics of ferromagnets and liquid crystals [4]. However, many interesting nonlinear systems are far from equilibrium, and cannot be described by a free energy functional, i.e., the governing order-parameter equation is not of gradient type. Nevertheless, the symmetries of the system are frequently preserved even far from the gradient limit. Thus pairs of equivalent solutions exist, but the net force acting on an interface between them is not necessarily zero. Examples of a transition from resting to moving fronts were found in the complex parametrically driven Ginzburg-Landau equation [2,5], in an activator-inhibitor reaction-diffusion system [3], and in optical parametric oscillators [8]. The corresponding bifurcation is often referred to as the nonequilibrium Ising-Bloch (NIB) transition.

The main result of this paper is to obtain a rather simple but universal criterion for the onset of a NIB transition and a generic amplitude equation, which can be applied to all known cases. To this end we identify symmetries, which are essential for a NIB transition, but more general than those presented in Refs. [2–5]. The investigations are based on the crucial finding that for each nonlinear nonequilibrium system a bifurcation of a resting solitary wave to a moving one is linked to an internal mode which comes into *exact* coincidence with the translational mode of the system. In this context we compare dissipative (gradient and nongradient) and Hamiltonian systems. The formalism and bifurcation scenario described in this work is rather general. For definiteness, we study an example from nonlinear optics where the formation and control of localized structures and fronts recently attracted a considerable deal of interest [6,7].

Due to the availability of materials with large second order suceptibilities, much attention has been given to parametric processes [8–17]. Here we study intracavity type-II second harmonic generation in a planar waveguide resonator, where two orthogonally polarized pump photons at frequency ω generate one signal photon at 2ω. The normalized set of equations for the slowly varying envelopes of the two orthogonally polarized fundamental harmonic fields $A_{1,2}$ (FH1, FH2) and of the second harmonic field B (SH) reads, in the mean field limit, as [16]

$$(i\partial_t + \partial_x^2 + \Delta_A + i)A_{1,2} + A_{2,1}^* B = E,$$

$$\left(i\partial_t + \frac{1}{2}\partial_x^2 + \Delta_B + i\gamma\right)B + A_1 A_2 = 0,$$

(1)

where ∂_x^2 describes diffraction, and t is the dimensionless time. The incident field E is a monochromatic plane wave with a polarization angle of 45°, thus driving both FH waves with the same intensity. The FH and SH fields are detuned by Δ_A and Δ_B from a resonator resonance, respectively. γ is the ratio of the photon lifetimes at the two frequencies [17]. A typical experimental configuration could consist of a 500-μm-thick KTP crystal sandwiched between two mirrors with 95% reflectivity for both fundamental and second harmonic waves. If the d_{31} coefficient is employed, and phase matching occurs at a certain tilt at 1.06 μm, one obtains the length and time scales as 30 μm and 110 ps. The driving intensity $|E|^2 = 1$ corresponds to about 50 kW/cm^2.

Equations (1) exhibit the translational symmetry and two discrete symmetries:

$$\mathcal{Z}:(A_1,A_2,B)\rightarrow(A_2,A_1,B), \quad \mathcal{P}:x\rightarrow -x. \quad (2)$$

$\mathcal{Z}$ allows for a pitchfork bifurcation of the stationary ($\partial_t = 0$), homogeneous ($\partial_x = 0$), and symmetric ($A_1 = A_2$) solutions [16]. There are two different resting fronts, i.e., heteroclinic trajectories, which connect equivalent states appearing after the pitchfork bifurcation. They are transformed into each other on applying $\mathcal{Z}$ [see Fig. 1(a)]. On the other side,
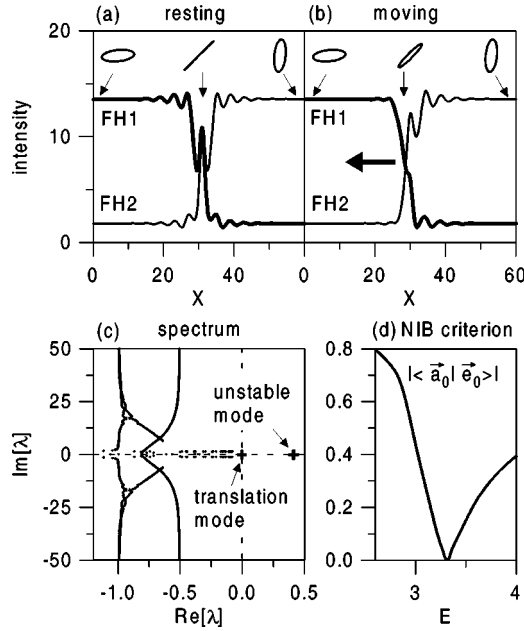
FIG. 1. Nonequilibrium Ising-Bloch transition of polarization fronts in intracavity second harmonic generation. (a) Ising front (unstable). (b) Bloch front (stable). Insets: FH polarization ellipse. (c) Spectrum of eigenvalues of the Ising front. (d) NIB criterion $|\langle \vec{a}_0 | \vec{e}_0 \rangle|$. Parameters: $\Delta_A = -0.7$, $\Delta_B = -4$, $E = 4$, and $\gamma = 0.5$.
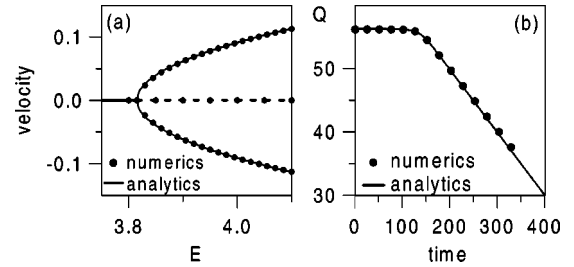

FIG. 2. Symmetry breaking bifurcation. (a) Bifurcation diagram for the front velocity close to the critical point; parameters: $\Delta_A = -0.31$, $\Delta_B = -4$, and $\gamma = 0.5$. (b) Evolution of an unstable resting to a stable moving front for $E = 4$.

each front is transformed into itself on applying $\hat{S} = \mathcal{Z}\mathcal{P}_0$, where the subscript ''0'' stresses that the reference frame for the spatial inversion was chosen at the point where $A_1 = A_2$. In what follows, fronts, which are invariant with respect to $\hat{S}$, are termed *Ising fronts*. Polarization fronts with similar symmetres were recently observed in a single feedback experiment with a sodium cell [18]. Ising fronts previously reported in Refs. [2,3,8] appear as particular cases of our definition. For example, the Ising front in the CPGL equation is antisymmetric with respect to the inversion operator $\mathcal{P}_0$. Therefore, the $\mathcal{Z}$ operator for the only occurring field A in the CPGL equation has to be chosen like $\mathcal{Z}: A \rightarrow -A$. Then this Ising front is symmetric with respect to $\hat{S}$, and all following considerations also hold for the CPGL model. Below, we will demonstrate that a transition from a resting symmetric front to a moving front is associated with a breaking of the $\hat{S}$ invariance, i.e., the symmetry operator $\hat{S}$ transforms a forward movint front into a backward moving front, and vice versa. These fronts are called *Bloch fronts*, and we term this bifurcation the NIB transition. Naturally, the eigenmode of the linearized operator driving this transition has to be antisymmetric to the $\hat{S}$ symmetry.

First we analyze the spectral properties of the resting polarization front. For the sake of numerical convenience, we switch to a real basis, rewriting Eqs. (1) in the general form

$$\partial_t \vec{u} - \vec{w}|_{\vec{u},p} = 0, \tag{3}$$

where $\vec{u}(x)$ is a six-component vector of the electrical fields $\vec{u}(x) = (\mathrm{Re}A_1, \mathrm{Im}A_1, \mathrm{Re}A_2, \mathrm{Im}A_2, \mathrm{Re}B, \mathrm{Im}B)^T$, and $\vec{w}$ is a

vector that is a nonlinear function of the fields and an arbitrary bifurcation parameter p. Obviously, resting fronts $\vec{u}_0$ obey the relation $\vec{w}|_{\vec{u}_0,p} = 0$. They are invariant with respect to the symmetry transformation $\hat{S}(\vec{u}_0(x)) = (u_3^0(-x), u_4^0(-x), u_1^0(-x), u_2^0(-x), u_5^0(-x), u_6^0(-x))^T = \vec{u}_0(x)$. Thus the eigenvectors $\vec{e}_n$ of the Jacobian $\partial_u \vec{w}|_{\vec{u}_0,p}$ (the first derivative of the nonlinear vector $\vec{w}$ with respect to the field $\vec{u}$ at the resting front $\vec{u}_0$) and the corresponding eigenvectors $\vec{a}_n$ of the adjoint Jacobian $(\partial_u \vec{w})^+|_{\vec{u}_0,p}$, can be either symmetric, $\hat{S}(\vec{e}_n) = \vec{e}_n$, or antisymmetric, $\hat{S}(\vec{e}_n) = -\vec{e}_n$. The resting front destabilizes if an eigenvalue, corresponding to any eigenvector $\vec{e}_n$, has a positive real part. An infinitesimal transverse translation of the resting front generates the nullvector $\vec{e}_0 = \partial_x \vec{u}_0$ (the translational mode) of $\partial_u \vec{w}|_{\vec{u}_0,p}$, i.e., $\partial_u \vec{w}|_{\vec{u}_0,p}(\vec{e}_0) = 0$. Both the nullvector $\vec{e}_0$ and the corresponding nullvector $\vec{a}_0$ ($\partial_u \vec{w}^+|_{\vec{u}_0,p}(\vec{a}_0) = 0$) of the adjoint Jacobian are antisymmetric.

We have found that at some critical value of the parameter $p = p_0$ (see the bifurcation diagrams in Fig. 2) the second-harmonic-generated (SHG) system undergoes a NIB transition. Each resting front transforms into a moving one, propagating either in positive or negative directions [see Fig. 1(b)]. By numerically solving the eigenvalue problem for $\partial_u \vec{w}|_{\vec{u}_0,p}$ we have found that the NIB transition is linked to a nontrivial and antisymmetric bound eigenmode of the Jacobian with a real eigenvalue [see Fig. 1(c)]. Therefore, SHG provides an example of a symmetry breaking of a symmetric solution due to an antisymmetric eigenvector. Additionally, we found that this eigenvector passes through the translational mode. In contrast in the gradient limit of the CPGL equation [2], the destabilization of an Ising front is initiated by an eigenvector which is *orthogonal* to the respective translational mode. However, we will show in the next paragraph that as soon as a nonlinear system leaves the gradient limit the NIB bifurcation is induced by an eigenvector passing *exactly* the translational mode.

Now we investigate this symmetry breaking close to the bifurcation point. We introduce a moving reference frame $\xi = x - \int_{t_0}^t v(t')dt'$ into Eq. (3), and obtain

$$\partial_t \vec{u} - v\,\partial_\xi \vec{u} - \vec{w}|_{\vec{u},p} = \vec{0}. \tag{4}$$

We assume the following scaling up to the third order around the critical point: $\vec{u}=\vec{u}_0+\epsilon\vec{X}+\epsilon^2\vec{Y}+\epsilon^3\vec{Z}$, $v=\epsilon v'+\epsilon^2 v''$ $+\epsilon^3 v'''$, $\partial_t=\epsilon^2\partial_T$, and $p=p_0+\epsilon^2 p_2$, with $\epsilon\ll 1$. In first order (ϵ^1) we obtain $-v'\partial_\xi\vec{u}_0-\partial_u\vec{w}|_{\vec{u}_0,p_0}(\vec{X})=0$, which can only be solved for

$$\langle\vec{a}_0|\vec{e}_0\rangle=0. \tag{5}$$

This relation is one of the key results of this paper, because it represents the criterion for the onset of a NIB transition. At the critical point one finds $\vec{X}=v'\vec{x}_1$, with $\partial_u\vec{w}|_{\vec{u}_0,p_0}(\vec{x}_1)=$ $-\vec{e}_0$. Thus in first order the shape of the front is disturbed by a component of velocity v' pointing in the direction of the antisymmetric vector $\vec{x}_1$. It can easily be shown that the vector $\vec{x}_1$ is linearly independent of all eigenvectors of the Jacobian $\partial_u\vec{w}|_{\vec{u}_0,p}$. Therefore, at the critical point two eigenvectors of the Jacobian must become degenerate to preserve the dimension of the vector space. In the case of a NIB transition a nontrivial bound state passes the translational mode. For symmetry reasons, in second order we obtain (ϵ^2) $\vec{Y}=p_2\vec{y}_1$ $+v'^2\vec{y}_2+v''\vec{x}_1$, with $\partial_u\vec{w}|_{\vec{u}_0,p_0}(\vec{y}_1)=-\partial_p\vec{w}|_{\vec{u}_0,p_0}$ and $\partial_u\vec{w}|_{\vec{u}_0,p_0}(\vec{y}_2)=-\partial_u^2\vec{w}|_{\vec{u}_0,p_0}(\vec{x}_1,\vec{x}_1)/2-\partial_\xi\vec{x}_1$. Finally the solvability condition applied to third order (ϵ^3) gives an amplitude equation for the velocity v' which represents the normal form for a symmetry breaking (pitchfork) bifurcation,

$$rp_2v'+sv'^3=\partial_T v', \tag{6}$$

where

$$r=\langle\vec{a}_0|\partial_\xi y_1+\partial_u^2\vec{w}|_{\vec{u}_0,p_0}(\vec{x}_1,\vec{y}_1)+\partial_{u,p}\vec{w}|_{\vec{u}_0,p_0}(\vec{x}_1)\rangle/\langle\vec{a}_0|\vec{x}_1\rangle$$

and

$$s=\langle\vec{a}_0|\partial_\xi y_2+\partial_u^2\vec{w}|_{\vec{u}_0,p_0}(\vec{x}_1,\vec{y}_2)$$

$$+\partial_u^3\vec{w}|_{\vec{u}_0,p_0}(\vec{x}_1,\vec{x}_1,\vec{x}_1)/6\rangle/\langle\vec{a}_0|\vec{x}_1\rangle.$$

In the case of type II intracavity second harmonic generation, $\partial_u^3\vec{w}$ vanishes. If the change of the input field $E=E_0$ $+\epsilon^2 E_2$ is the bifurcation parameter p, we also have $\partial_{u,p}\vec{w}$ $=0$ and $\partial_p\vec{w}=(0,1,0,1,0,0)$. Furthermore, numerics predict that the bifurcation is supercritical for the parameters selected. If $E_2<0$, there is only the trivial stationary solution $v'=0$, corresponding to a resting front. Beyond the critical point this branch becomes unstable and two new stable solutions $v'=\pm\sqrt{-rE_2/s}$ emerge, corresponding to the two counterpropagating fronts [see Fig. 2(a)]. The asymptotic expression for the spatial profiles of the fronts close to the bifurcation point can be calculated as $\vec{u}=\vec{u}_0+\epsilon v'\vec{x}_1$ $+\epsilon^2(p_2\vec{y}_1+v'^2\vec{y}_2+v''\vec{x}_1)$, where $\vec{x}_1$ is an antisymmetric sector and $\vec{y}_{1,2}$ are symmetric vectors. Therefore, the moving fronts themselves are neither symmetric nor antisymmetric. But the forward ($\vec{u}_{\text{for}}$) and backward moving fronts ($\vec{u}_{\text{back}}$) are related by $\hat{S}(\vec{u}_{\text{for}})=\vec{u}_{\text{back}}$.

Equation (6) explicitly shows that the moving polarization fronts relax exponentially to the final state, in contrast to the algebraic relaxation of the velocity of fronts propagating between stable and unstable states [19]. For example, the destabilization of a resting front for $E_2>0$ can be described by a direct integration of Eq. (6), to obtain $v'=$ $\pm\sqrt{-rE_2/(s+De^{2rE_2T})}$, where D is an integration constant. In order to compare the analytical and numerical findings, we display the dynamics of the integral of a single field component $Q(t)=\int_{x_1}^{x_2}u_1(x,t)dx$. For a sufficiently large interval (x_1,x_2), an analytical approximation $Q(t)=[u_1(x\rightarrow\infty)-u_1(x\rightarrow-\infty)]\int_{t_1}^t v'(t')dt'$ can be obtained and compared with the results of the numerical solution of Eq. (1) [see Fig. 2(b)].

It is worthwhile to have a closer look at the threshold condition $\langle\vec{a}_0|\vec{e}_0\rangle=0$. It is obvious that for Galilean symmetry, or in Hamiltonian systems with translational invariance, $\langle\vec{a}_0|\vec{e}_0\rangle$ is always zero, and thus the velocity of localized solutions is an arbitrary parameter [20]. This signals a noncritical, i.e., parameter independent, double degeneracy of the zero eigenvalues corresponding to $\vec{e}_0$; see Ref. [21], and references therein. Conversely, if the above properties do not hold, then any initially introduced velocity converges to a definite value. But the above analysis discloses that even in a dissipative system without Galilean invariance, the velocity can be considered as a parameter of the solution, provided that the scalar product $\langle\vec{a}_0|\vec{e}_0\rangle$ vanishes for some critical values of the system parameters and $\langle\vec{a}_0|\vec{e}_0\rangle$ remains sufficiently small. Now, however, the velocity is not an arbitrary parameter, but rather an order parameter obeying the normal form [Eq. (6)]. Exactly at the critical point the dissipative system behaves in a Hamiltonian-like manner. In contrast, in gradient systems criterion (5) is never fulfilled, because the corresponding Jacobian is self-adjoint. Consequently the eigenvector which causes the instability is orthogonal to the translational mode; therefore, the front remains at rest. But if a small nongradient term is added to a gradient equation the scalar product $|\langle\vec{a}_0|\vec{e}_0\rangle|$ drops to zero at the bifurcation point, accompanied by a rapid transition of the respective eigenvector to the translational mode. In view of this scenario, Ising-Bloch transitions appear very peculiar in gradient systems.

It is important to stress that there is no need to study the full spectral problem for $\partial_u^+\vec{w}|_{\vec{u}_0,p}$ to find $\vec{a}_0$. It suffices to solve the linear boundary value problem $\partial_u\vec{w}^+|_{\vec{u}_0,p}(\vec{a}_0)=0$. Then, applying one of the standard algorithms to minimize $|\langle\vec{a}_0|\vec{e}_0\rangle|$, one can easily identify whether there is a NIB transition for a given solution in a nonlinear system [see Fig. 1(d)]. The suggested theoretical approach can be straightforwardly extended to other dissipative models which exhibit a transition from resting to moving pulses [22].

In conclusion, based on symmetry arguments, we have introduced generalized definitions of both Ising and Bloch fronts. A universal criterion and a generic equation for the onset of a nonequilibrium Ising-Bloch transition, which relies on the coincidence of an eigenmode and the translational mode at the bifurcation point, have been derived. Dissipative (gradient and nongradient) and Hamiltonian systems have been compared in that context.

[1] M.C. Cross and P.C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993); W. van Saarloos and P.C. Hohenberg, Physica D **56**, 303 (1992).

[2] P. Coullet *et al.*, Phys. Rev. Lett. **65**, 1352 (1990).

[3] A. Hagberg *et al.*, Phys. Rev. Lett. **76**, 427 (1996); Phys. Rev. E **55**, 4450 (1997); A. Hagberg and E. Meron, Phys. Rev. Lett. **78**, 1166 (1997).

[4] L.N. Bulaevsky and V.L. Ginzburg, Zh. Éksp. Teor. Fiz. **45**, 772 (1963) [Sov. Phys. JETP **18**, 530 (1964)]; J.M. Gilli, M. Morabito, and T. Frisch, J. Phys. II **4**, 319 (1994).

[5] G. de Valcarcel and K. Staliunas (unpublished).

[6] C.O. Weiss *et al.*, e-print patt-sol/0003015.

[7] V.B. Taranenko, K. Staliunas, and C.O. Weiss, Phys. Rev. Lett. **81**, 2236 (1998); S. Coen *et al.*, *ibid.* **83**, 2328 (1999); V.B. Taranenko and C.O. Weiss, e-print patt-sol/0003004.

[8] G. Izus, M. San Miguel, and M. Santagiustina, Opt. Lett. **25**, 1454 (2000).

[9] S. Longhi, Phys. Scr. **56**, 611 (1997).

[10] K. Staliunas and V.J. Sánchez-Morcillo, Phys. Rev. A **57**, 1454 (1998).

[11] S. Trillo, M. Haelterman, and A. Sheppard, Opt. Lett. **22**, 970 (1997); T.N. Kutz *et al.*, J. Opt. Soc. Am. B **16**, 1936 (1999).

[12] G.L. Oppo, A. Scroggie, and W.J. Firth, J. Opt. B: Quantum Semiclass. Opt. **1**, 133 (1999).

[13] M. Le Berre *et al.*, J. Opt. B: Quantum Semiclass. Opt. **1**, 153 (1999).

[14] M. Tlidi, P. Mandel, and R. Lefever, Phys. Rev. Lett. **81**, 979 (1998); M. Tlidi *et al.*, Opt. Lett. **25**, 487 (2000).

[15] M. Taki *et al.*, J. Opt. Soc. Am. B **17**, 997 (2000); M. Santagiustina *et al.*, Opt. Lett. **23**, 1167 (1998).

[16] U. Peschel *et al.*, Phys. Rev. E **58**, R2745 (1998).

[17] D.V. Skryabin, A.R. Champneys, and W.J. Firth, Phys. Rev. Lett. **84**, 463 (2000).

[18] E. Große Westhoff *et al.*, J. Opt. B: Quantum Semiclass. Opt. **2**, 386 (2000).

[19] U. Ebert and W. van Saarloos, Phys. Rev. Lett. **80**, 1650 (1998); C. Storm *et al.*, e-print patt-sol/9908007.

[20] D.E. Pelinovsky, A.V. Buryak, and Y.S. Kivshar, Phys. Rev. Lett. **75**, 591 (1995).

[21] D.V. Skryabin, Physica D **139**, 186 (2000).

[22] V.V. Afanassjev, N. Akhmediev, and J.M. Soto-Crespo, Phys. Rev. E **53**, 1931 (1996); S.V. Fedorov *et al.*, *ibid.* **61**, 5814 (2000).